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Infinitely Many Nonhomogeneous Conditions

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A general formula is given for the index of a linear manifold in Banach space. This is expressed in terms of the dimensions of linear manifolds. A fundamental result is established, called a Boundary-Form Formula, for linear manifolds. This is then used to study the existence of the solutions of boundary value problems for linear manifolds subject to finite or infinite nonhomogeneous boundary conditions. An application is made to study the deficiency index of a second-order ordinary differential operator by dividing an interval into infinite subintervals. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let $T_0 \subset T_1$ be closed linear manifolds in the direct sum of Banach spaces such that the quotient space T_1/T_0 is isomorphic to an infinite-dimensional, separable Hilbert space. The main purpose of this paper is threefold:

First, we give index formulas for intermediate closed linear manifolds T between T_0 and T_1. These are expressed in terms of the dimensions of the null spaces of "maximal" subspaces and the numbers of the boundary conditions defining the intermediate subspaces and the ones defining their adjoints. In the case when the intermediate space is the graph of an elliptic operator on a compact oriented differential manifold, Atiyah and Singer [1] gave index formulas which are expressed by a Chern characteristic and a Todd class. However, our different index-formulas for general linear manifolds are elementary and are motivated from ordinary differential operator.

Second, we express a natural pairing restricted to the direct sum of maximal subspaces in terms of those functionals defining any given intermediate subspace and the ones defining its adjoint. This fundamental representation, called a Boundary-Form Formula, is then used to study the existence of the solutions to boundary value problems for linear manifolds subject to finite or infinite nonhomogeneous boundary conditions. This is an infinite-dimensional generalization of its finite-dimensional analogue discussed in [10]. Such problems occur naturally from ordinary differential

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operator if we define a solution to be piecewise continuous at given infinitely many points. It is hoped that the theory will find its application in partial differential operator as well as linear control theory.

Third, we apply the theory to a second-order ordinary differential operator, in particular to a deficiency index problem. This is done by subdividing an interval into infinitely many pieces and by viewing a continuity condition as a boundary condition.

The paper is in essence expository in style, and the main tools used are the adjoint theory developed in [8, 9] and Besselian–Hilbertian basis.

We now fix some notations. For a Banach space X, the Banach space of all continuous conjugate linear functions from X is denoted by X^*. Let T be a linear manifold (vector space) contained in the direct sum $X_1 \oplus X_2$ of Banach spaces X_1 and X_2. Then

$$\text{the adjoint (Aren's), } T^*, \text{ of } T$$
$$\equiv \{\{b_2, b_1\} \in X_2^* \oplus X_1^* \mid \bar{b}_2(a_2) - \bar{b}_1(a_1) = 0$$

for all $\{a_1, a_2\} \in T\}$,

$$\text{Null } T \equiv \{a_1 \in X_1 \mid \{a_1, 0\} \in T\},$$
$$\text{Domain } T \equiv \{a_1 \in X_1 \mid \{a_1, a_2\} \in T \text{ for some } a_2 \in X_2\},$$
$$\text{Range } T \equiv \{a_2 \in X_2 \mid \{a_1, a_2\} \in T \text{ for some } a_1 \in X_1\}.$$

If $\tilde{T}$ is a linear manifold in $X_2^* \oplus X_1^*$, then

$$\text{the preadjoint (Aren's), } {}^*\tilde{T}, \text{ of } \tilde{T}$$
$$\equiv \{\{a_1, a_2\} \in X_1 \oplus X_2 \mid \bar{b}_2(a_2) - \bar{b}_1(a_1) = 0 \text{ for all } \{b_2, b_1\} \in \tilde{T}\}.$$

For more information on adjoints, see Coddington and Dijksma [2], where other references may also be found. The algebraic sum of two linear manifolds M_1 and M_2 is denoted by $M_1 \dot{+} M_2$. Let D_1 and D_2 be $m \times n$ and $p \times q$ constant matrices. The conjugate transpose of D_1 is denoted by D_1^* and the transpose of D_1 is denoted by D_1^{t}. The supscript t will also be used to denote the transpose of a matrix whose elements are in a vector space.

In the case when D_1 is a finite matrix, $D_1 \oplus D_2$ will denote the $(m+p) \times (n+q)$ matrix obtained by joining the lower right corner of D_1 to the upper left corner of D_2. The Hilbert space of all $1 \times m$ complex *matrices* a with $aa^* < \infty$ is denoted by l_2^m. When $m = \infty$, the simpler notation for l_2^m is l_2. Suppose that D_1 satisfies the condition: For each $a \in l_2^m$, aD_1 exists, and $aD_1 \in l_2^m$. Such a matrix is called a Hilbert matrix by Cooke [3]. See [8] for more references. D_1 is termed a non-singular Hilbert matrix if it is a Hilbert matrix and $m = n$ and the map $a \to aD_1$ is one-to-one from l_2^m onto

l_2^m. The inverse of a non-singular Hilbert matrix D_1 is denoted by D_1^{-1}. Suppose that P is a $m \times n$ Hilbert matrix. Then

$$\begin{aligned}\text{Domain}(\cdot P) &\equiv l_2^m,\\ \text{Range}(\cdot P) &= \{aP \mid a \in l_2^m\},\\ \text{Null}(\cdot P) &= \{a \in l_2^m \mid aP = O_{1\times n}\}.\end{aligned}$$

The closed linear subspace of l_2^n generated by all the *rows* of P is denoted by $\langle P\rangle$. P is called normalized if it satisfies one of the following: If $m < \infty$, then the rows of P are linearly independent in l_2^n; if $m = \infty$, then $PP^* = I_\infty$, the $\infty \times \infty$ identity matrix. Finally, the set of positive integers is denoted by N.

2. Index and Nonhomogeneous Conditions

Let X_1 and X_2 be Banach spaces over the complex field C. Let $T_0 \subset T_1$ be arbitrary, but fixed closed linear manifolds in the direct sum $X_1 \oplus X_2$ such that T_1/T_0 is isomorphic to l_2, and Null T_1 and Null T_0^* are isomorphic to separable Hilbert spaces. The spaces T_0 and T_1^* are called minimal subspaces and T_1 and T_0^* are called maximal subspaces. We also assume that T_0 is complemented in T_1. Thus, there exists a bounded linear operator B from T_1 onto l_2 whose kernel is T_0, and there exists a w^*-continuous linear operator from T_0^* onto l_2 whose kernel is T_1^*. It then follows that there exists a unique $\infty \times \infty$ non-singular Hilbert matrix C depending only on B and B^+ such that

$$\bar{b}_2(a_2) - \bar{b}_1(a_1) = iB(a)\, C(B^+(b))^* \tag{2.1}$$

for all $a = \{a_1, a_2\} \in T_1$, $b = \{b_2, b_1\} \in T_0^*$.

For the existence of such B, B^+ and C, see [9, 8]. *Throughout this paper unless otherwise mentioned,* T_0, T_1, B, B^+ *and* C *will be as the above.*

It is shown in [9] (also in [8] for a Hilbert space case) that a linear manifold T in $X_1 \oplus X_2$ is a closed one between T_0 and T_1 if, and only if,

$$T = \{a \in T_1 \mid P(B(a))^* = O_{m\times 1}\} \tag{2.2}$$

for a $m \times \infty$ Hilbert matrix P, and a linear manifold $\tilde{T}$ in $X_2^* \oplus X_1^*$ is a w^*-closed one between T_1^* and T_0^* if, and only if,

$$T = \{b \in T_0^* \mid \tilde{P}(B^+(b))^* = O_{\tilde{m}\times 1}\}. \tag{2.3}$$

Moreover, $T^* = \tilde{T}$ (or equivalently ${}^*\tilde{T} = T$) if, and only if,

$$\langle \tilde{P}C^{-1}\rangle = l_2 \ominus \langle P\rangle. \tag{2.4}$$

In particular,

$$T^* = \{b \in T_0^* \mid B^+(b)\, C^* \in \langle P \rangle\},$$
$$*\tilde{T} = \{a \in T_1 \mid B(a) C \in \langle \tilde{P} \rangle\}.$$

If P is normalized, then

$$T^* = \{b \in T_0^* \mid (I_\infty - P^*(PP^*)^{-1}P)\, C(B^+(b))^* = O_{\infty \times 1}\},$$

and if $\tilde{P}$ is normalized, then

$$*\tilde{T} = \{a \in T_1 \mid (I_\infty - \tilde{P}^*(\tilde{P}\tilde{P}^*)^{-1}\tilde{P})\, C^*(B(a))^* = O_{\infty \times 1}\}.$$

If P is normalized, then $m = \dim\langle P \rangle$. For this reason the number m is called the number of boundary conditions for T when P is normalized. The m functions of $a \in T_1$ in the definition of $P(B(a))^*$ are called boundary functions for T defined in (2.2).

The following shows that the natural pairing

$$\bar{b}_2(a_2) - \bar{b}_1(a_1)$$

for $(X_1 \oplus X_2) \oplus (X_2^* \oplus X_1^*)$ restricted to $T_1 \oplus T_0^*$ is decomposed by the boundary functionals defining any given closed linear manifold T and the boundary functionals defining T^*. This is fundamental to our subsequent development. It's finite-dimensional analogue was proved in Theorem 3.1 of [10].

THEOREM 1 (Boundary-Form Formula). *Let P be a $m \times \infty$ normalized Hilbert matrix. Then PP^*, $\tilde{P}C^{-1}(\tilde{P}C^{-1})^*$, $PC^{*-1}(PC^{*-1})^*$ and $\tilde{P}\tilde{P}^*$ are $m \times m$, $\tilde{m} \times \tilde{m}$, $m \times m$ and $\tilde{m} \times \tilde{m}$ non-singular Hilbert matrices.*

If we assume further that

$$\langle \tilde{P}C^{-1} \rangle = l_2 \ominus \langle P \rangle,$$

then we have the following:

$$\text{(I)}\quad \bar{b}_2(a_2) - \bar{b}_1(a_1) = iB(a)\, P^*(PP^*)^{-1}PC(B^+(b))^* + iB(a)\, C^{*-1}\tilde{P}^*(\tilde{P}C^{-1}C^{*-1}\tilde{P}^*)^{-1}\tilde{P}(B^+(b))^*$$

for all $a = \{a_1, a_2\} \in T_1$, $b = \{b_2, b_1\} \in T_0^*$.

$$\text{(II)}\quad \bar{b}_2(a_2) - \bar{b}_1(a_1) = iB(a)\, P^*(PC^{*-1}C^{-1}P^*)^{-1}PC^{*-1}(B^+(b))^* + iB(a)\, C\tilde{P}^*(\tilde{P}\tilde{P}^*)^{-1}\tilde{P}(B^+(b))^*$$

for all $a = \{a_1, a_2\} \in T_1$, $b = \{b_2, b_1\} \in T_0^*$.

Proof. First we will show that PP^* and $\tilde{P}C^{-1}(\tilde{P}C^{-1})^*$ are non-singular. This will follow from the following lemma:

LEMMA 2. *Suppose that E is $\infty \times \infty$ non-singular Hilbert matrix. Let A and F be $m \times \infty$ normalized Hilbert matrices such that $\langle AE \rangle = \langle F \rangle$.*
Then AEF^* is a $m \times m$ non-singular Hilbert matrix.

Proof of the Lemma. If $m < \infty$, the result is clear. Assume $m = \infty$. Let $\tilde{F}$ be a $\tilde{m} \times \infty$ Hilbert matrix such that $\tilde{F}\tilde{F}^* = I_{\tilde{m}}$, the $\tilde{m} \times \tilde{m}$ identity matrix, and $\langle \tilde{F} \rangle = l_2 \ominus \langle F \rangle$. Suppose that $\alpha AEF^* = O_{1 \times \infty}$ for some $\alpha \in l_2$. Then $\alpha AE \in \langle \tilde{F} \rangle$, and so $\alpha AE = \beta \tilde{F}$ for some $\beta \in l_2^{\tilde{m}}$. Then

$$\beta = \alpha AE\tilde{F}^* = 0.$$

and so $\alpha = 0$. This shows that the map

$$\alpha \to \alpha AEF^* \qquad \text{for} \quad \alpha \in l_2$$

is one-to-one. Suppose $\beta \in l_2$. Then

$$\beta FE^{-1}A^* = \beta FE^{-1}A^*,$$

and so

$$\beta FE^{-1}A^*A - \beta FE^{-1} \in l_2 \ominus \langle A \rangle.$$

It follows that

$$\beta FE^{-1}A^*A = \beta FE^{-1}.$$

Thus

$$\beta FE^{-1}A^*AEF^* = \beta FF^* = \beta.$$

This shows that the map mentioned above is onto. Thus AEF^* is non-singular.

We now return to the proof of the theorem. Let Q and $\tilde{Q}$ be the $m \times \infty$ and $\tilde{m} \times \infty$ Hilbert matrices such that

$$QQ^* = I_m, \qquad \tilde{Q}\tilde{Q}^* = I_{\tilde{m}},$$
$$\langle P \rangle = \langle Q \rangle, \qquad \langle \tilde{P}C^{-1} \rangle = \langle \tilde{Q} \rangle. \tag{2.5}$$

Then

$$\langle Q \rangle = l_2 \ominus \langle \tilde{Q} \rangle. \tag{2.6}$$

Since $\tilde{Q}Q^* = O_{\tilde{m}\times m}$, it follows from (2.6) together with [11, Proposition 1.3 that

$$Q^*Q + \tilde{Q}^*\tilde{Q} = I_\infty. \tag{2.7}$$

Thus for $a = \{a_1, a_2\} \in T_1$, $b = \{b_2, b_1\} \in T_0^*$,

$$B(a)\, C(B^+(b))^* = B(a)\, Q^*QC(B^+b))^* + B(a)\, \tilde{Q}^*\tilde{Q}C(B^+(b))^*. \tag{2.8}$$

Now, by (2.5) and Lemma 2, the Hilbert matrices PQ^* and $\tilde{P}C^{-1}\tilde{Q}^*$ are non-singular. Thus

$$Q = (PQ^*)^{-1}P,\ \tilde{Q} = (\tilde{P}C^{-1}\tilde{Q}^*)^{-1}\tilde{P}C^{\ \ 1}.$$

It follows that

$$\begin{aligned} Q^*Q &= P^*(PQ^*QP^*)^{-1}P, \\ \tilde{Q}^*\tilde{Q} &= (\tilde{P}C^{-1})^*(\tilde{P}C^{-1}\tilde{Q}^*\tilde{Q}C^{*-1}\tilde{P}^*)^{-1}\tilde{P}C^{-1}. \end{aligned} \tag{2.9}$$

Since

$$I_m = (PQ^*)^{-1}PP^*(QP^*)^{-1},$$

and

$$\begin{aligned} I_{\tilde{m}} &= (\tilde{P}C^{-1}\tilde{Q}^*)^{-1}\tilde{P}C^{-1}C^{*-1}\tilde{P}^*(\tilde{Q}C^{*-1}\tilde{P}^*)^{-1}, \\ (PP^*)^{-1} &= (PQ^*QP^*)^{-1}, \\ (\tilde{P}C^{-1}C^{*-1}\tilde{P}^*)^{-1} &= (\tilde{P}C^{-1}\tilde{Q}^*\tilde{Q}C^{*-1}\tilde{P}^*)^{-1}. \end{aligned} \tag{2.10}$$

Thus (I) follows from (2.8) together with (2.9) and (2.10) and (2.1). Part (II) can be proved in a similar way. This completes the proof.

If R is a $m \times n$ Hilbert matrix, then we define Rank R to be $\dim\langle R\rangle$ if $m < \infty$, and to be ∞ if infinitely many rows of R are linearly independent in l_2.

The idea of the proof of the following is contained in [6, p. 325]. We will give a proof for completeness.

LEMMA 3. *For a Hilbert matrix R,*

$$\text{Rank } R = \text{Rank } R^*.$$

Proof. Assume that R is $m \times n$. Let W be the map $\alpha \to \alpha R$ for $\alpha \in l_2^m$. Let p^j be the jth column of R. Then

$$\text{Rank } R = \dim(l_2^m \ominus \text{Ker } W).$$

But

$$\text{Ker } W = (\text{linear span } \{(p^j)^* \mid 1 \leqslant j \leqslant n\})^{\perp}.$$

It follows that

$$\begin{aligned}\text{Rank } R &= \dim(\text{linear span } \{(p^j)^* \mid 1 \leqslant j \leqslant n\})^{c} \\ &= \text{Rank } R^*,\end{aligned}$$

where the superscript c denotes the closure. This completes the proof.

Let

$$n = \dim \text{Null } T_1, \qquad n^* = \dim \text{Null } T_0^*.$$

Then there exists a Besselian–Hilbertian basis $\{\phi_j \mid 1 \leqslant j \leqslant n\}$ for Null T_1, and there exists a Besselian–Hilbertian basis $\{\psi_j \mid 1 \leqslant j \leqslant n^*\}$ for Null T_0^*. Let us define $n \times \infty$ and $n^* \times \infty$ matrices (necessarily Hilbert matrices) G and $\tilde{G}$ by

$$G = \begin{pmatrix} B(\{\phi_1, 0\}) \\ \vdots \\ B(\{\phi_n, 0\}) \end{pmatrix}, \qquad \tilde{G} = \begin{pmatrix} B^+(\{\psi_1, 0\}) \\ \vdots \\ B^+(\{\psi_{n^*}, 0\}) \end{pmatrix}. \tag{2.11}$$

Let Φ be the $1 \times n$ matrix

$$[\phi_1, ..., \phi_n],$$

and let Ψ be the $1 \times n^*$ matrix

$$[\psi_1, ..., \psi_{n^*}].$$

The following was proved in [10].

Proposition 4. *Let P and $\tilde{P}$ be $m \times \infty$ and $\tilde{m} \times \infty$ normalized Hilbert matrices satisfying* (2.4).

Let T be as (2.2). *Then we have the following*:

(I) *Null* $T = \{\alpha \Phi^t \mid \alpha \in l_2^n, \alpha G P^* = O_{1 \times m}\}$.

(II) *Null* $T^* = \{\alpha \Psi^t \mid \alpha \in l_2^{n^*}, \alpha \tilde{G} \tilde{P}^* = O_{1 \times \tilde{m}}\}$.

(III) *Null* $T_1 =$ *Null* $T \dotplus \langle P G^* \Phi^t \rangle$ *(direct sum)*,
Null $T_0^* =$ *Null* $T^* \dotplus \langle \tilde{P} \tilde{G}^* U^t \rangle$ *(direct sum)*.

Here $\langle P G \Phi^t \rangle$ is the closed subspace of X_1 generated by all the rows of $P G \Phi^t$.

For a linear manifold M in $X_1 \oplus X_2$, the index of M is defined by

$$\dim \text{Null } M - \dim \text{Null } M^*$$

whenever both numbers are not infinite at the same time. In the case when M is the graph of an operator, this definition coincides with the usual definition of the index of an operator. When M is the graph of an elliptic operator on a compact oriented differential manifold (in this case Null M and Null M^* are finite dimensional), Atiyah and Singer [1] gave a formula for the index of M. It involves a Chern character and a Todd class. We will give an entirely different (and elementary) approach to the index formulas, this time for linear manifolds when the dimensions of null spaces may be infinite. Our main emphasis is to relate the dimensions of Null M and Null M^* with the dim T_0^*, dim Null T_1 and the numbers of the boundary conditions defining M and M^*.

This approach turned out to be quite useful for ordinary differential operators. The finite-dimensional case was considered in [10].

THEOREM 5. *Let P be a $m \times \infty$* normalized *Hilbert matrix, and $\tilde{P}$ be a $\tilde{m} \times \infty$* normalized *Hilbert matrix. Assume that*

$$\langle \tilde{P} \rangle = l_2 \ominus \langle PC^{*\,-1} \rangle.$$

(Thus $\tilde{m} = \dim\langle \tilde{P} \rangle$.) Let

$$T = \{a \in T_1 \mid P(B(a))^* = O_{m \times 1}\}.$$

Then we have the following:

(I) $\dim \mathit{Null}\, T_1 + \dim \mathit{Null}(\cdot PG^*) = \dim \mathit{Null}\, T + m,$

$\dim \mathit{Null}\, T_0^* + \dim \mathit{Null}(\cdot \tilde{P}\tilde{G}^*) = \tilde{m} + \dim \mathit{Null}\, T^*.$

(II) *Assume that Range T_0^* is w^*-dense in X_1^*, and Range T_1 is dense in X_2. Then*

$$\dim \mathit{Null}\, T + \mathit{Rank}\, \tilde{G}\tilde{P}^* \leqslant \tilde{m},$$

$$\dim \mathit{Null}\, T^* + \mathit{Rank}\, GP^* \leqslant m,$$

and the following (i) *and* (ii) *are equivalent.*

(i) $l_2 = \{B(a)C + B^+(b) \mid a \in \mathit{Null}\, T_1 \oplus \{0\},$
$\qquad b \in \mathit{Null}\, T_0^* \oplus \{0\}\}.$

(ii) $\{x \in l_2^{\tilde{m}} \mid xPG^* = O_{1\times n^*}\}$
$\qquad = \{B(a)\, C^{*-1}\tilde{P}(\tilde{P}C^{-1}C^{*-1}\tilde{P}^*)^{-1} \mid a \in \mathit{Null}\, T \oplus \{0\}\}$

and

$\{x \in l_2^{m} \mid xPG^* = O_{1\times n}\}$
$\qquad = \{B^+(b)\, C^*P^*(PP^*)^{-1} \mid b \in \mathit{Null}\, T^* \oplus \{0\}\}.$

(III) *Assume that Range* T_0^* *is* w^**-dense in* X_1^* *and Range* T_1 *is dense in* X_2.

If (i) *or* (ii) *above holds, then we have the following* (i), (ii) *and* (iii):

(i) $\dim \mathit{Null}\, T_1 + \dim \mathit{Null}\, T_0^* = \infty$.

(ii) $\dim \mathit{Null}\, T + \dim \mathit{Null}\, T_0^* = \dim \mathit{Null}\, T^* + \tilde{m}$

and

$$\dim \mathit{Null}\, T^* + \dim \mathit{Null}\, T_1 = \dim \mathit{Null}\, T + m.$$

(iii) $\dim \mathit{Null}\, T = \dim \mathit{Null}(\cdot \tilde{P}\tilde{G}^*)$

and

$$\dim \mathit{Null}\, T^* = \dim \mathit{Null}(\cdot PG^*).$$

Proof. (I) By Proposition 4,

$$\dim \text{Null } T_1 = \dim \text{Null } T + \text{Rank } PG^*.$$

But

$$m = \dim \text{Null}(\cdot PG^*) + \text{Rank } GP^*.$$

It follows that

$$\dim \text{Null } T_1 + \dim \text{Null}(\cdot PG^*) = \dim \text{Null } T + m$$

as

$$\text{Rank } PG^* = \text{Rank } GP^*.$$

The second part of (I) can be proved in a similar way.

(II) Define operators U^+ and V by

$$U^+(b) = B^+(b)\, C^* P^* (PP^*)^{-1}, \qquad b \in \text{Null } T_0^* \oplus \{0\},$$
$$V(a) = B(a)\, C^{*-1} \tilde{P}^* (\tilde{P} C^{-1} C^{*-1} \tilde{P}^*)^{-1}, \qquad a \in \text{Null } T_1 \oplus \{0\}.$$

Then it follows from Theorem 1 that

$$\begin{aligned} 0 &= B(a)\, C (B^+(b))^* \\ &= B(a)\, P^* (U^+(b))^* + V(a)\, \tilde{P} (B^+(b))^* \end{aligned} \tag{2.12}$$

for $a \in \text{Null } T_1 \oplus \{0\}$, $b \in \text{Null } T_0^* \oplus \{0\}$. Then, together with the definitions of G and $\tilde{G}$,

$$\begin{aligned} U^+(b)(GP^*)^* &= O_{1\times n}, \qquad b \in \text{Null } T^* \oplus \{0\}, \\ V(a)(\tilde{G}\tilde{P}^*)^* &= O_{1\times n}, \qquad a \in \text{Null } T \oplus \{0\}. \end{aligned} \tag{2.13}$$

It follows that the restriction, $U^+_{T^*}$, of U^+ to Null $T^* \oplus \{0\}$ defines a bounded linear operator from Null $T^* \oplus \{0\}$ into Null$(\cdot PG^*)$, and the restriction, V_T, of V to Null $T \oplus \{0\}$ also defines a bounded operator into Null$(\cdot \tilde{P}\tilde{G}^*)$. Since dim Null $T_0 = 0$, V_T is one-to-one. Since dim Null $T_1^* = 0$, $U^+_{T^*}$ is one-to-one. Since

$$\text{Rank } PG^* = \text{Rank } GP^*,$$

$$\text{Rank } \tilde{P}\tilde{G}^* = \text{Rank } \tilde{G}\tilde{P}^*,$$

and separable Hilbert spaces differ only by their dimensions, it follows that there exist an isomorphism F_1 from $\langle PG^* \rangle$ onto $\langle GP^* \rangle$ and an isomorphism F_2 from $\langle \tilde{P}\tilde{G}^* \rangle$ onto $\langle \tilde{G}\tilde{P}^* \rangle$. Let us define operators A_k, A_k^+ as follows:

$$A_1(x) = B(\{x\Phi^t, 0\}), \qquad x \in l_2^n,$$

$$A_1^+(y) = B^+(\{y\Psi^t, 0\})\, C^*, \qquad y \in l_2^{n'},$$

$$A_2(x) = \{x\Phi^t, 0\}, \qquad x \in \text{Null}(\cdot GP^*),$$

$$A_2^+ = \{y\Psi^t, 0\}, \qquad y \in \text{Null}(\cdot \tilde{G}\tilde{P}^*),$$

$$A_3(x) = x\tilde{P}, \qquad x \in l_2^{\tilde{m}},$$

$$A_3^+(y) = yPC^{*-1}, \qquad y \in l_2^m.$$

Since Null T_0 and Null T_1^* are trivial space, it follows from the definitions of Φ, Ψ, B and B^+ that A_1 and A_1^+ are isomorphisms onto. Clearly, A_2, A_2^+, A_3, A_3^+ are isomorphisms onto. Therefore we have the following diagram:

$$\begin{tikzcd}
\{B(a) \mid a \in \text{Null } T_1 \oplus \{0\}\}, & \{B^+(b)\, C^* \mid b \in \text{Null } T_0^* \oplus \{0\}\}, \\
l_2^n = \text{Null}(\cdot GP^*) \dotplus \langle PG^* \rangle, \arrow[u, "A_1"] \arrow[d, "A_2"] \arrow[ddr, "F_1"] & \langle \tilde{P}\tilde{G}^* \rangle \dotplus \text{Null}(\cdot \tilde{G}\tilde{P}^*) = l_2^{n'}, \arrow[u, "A_1^+"'] \arrow[d, "A_2^+"] \arrow[ddl, "F_2"'] \\
\text{Null } T \oplus \{0\}, \arrow[d, dashed, "V_T"] & \text{Null } T^* \oplus \{0\} \arrow[d, dashed, "U^+_{T^*}"] \\
\text{Null}(\cdot \tilde{P}\tilde{G}) \dotplus \langle \tilde{G}\tilde{P}^* \rangle = l_2^{\tilde{m}}, \arrow[d, "A_3"] & l_2^m = \langle GP^* \rangle \dotplus \text{Null}(\cdot PG^*), \arrow[d, "A_3^+"] \\
\langle \tilde{P} \rangle, & \langle PC^{*-1} \rangle,
\end{tikzcd}$$

where all the algebraic sums, $\dotplus$, are orthogonal, the unbroken arrows indicate isomorphisms onto, and the broken arrows indicate one-to-one continuous maps, but may not be onto.

The first two parts of (II) are immediate by the diagram. We will prove the last part.

Assume (II)(ii). Then all the maps in the diagram are onto. Thus (II)(i) holds as l_2 is the orthogonal direct sum of $\langle \tilde{P} \rangle$ and $\langle PC^{*-1} \rangle$, and

$$\{B(a) \mid a \in \text{Null } T_1 \oplus \{0\}\}$$

and

$$\{B^+(b)\, C^* \mid b \in \text{Null } T_0^* \oplus \{0\}\}$$

are closed linear manifolds orthogonal to each other.

We now assume (II)(i). We must show that V_T and $U_{T^*}^+$ are onto. Take any $x \in l_2^{\tilde{m}}$ such that $x\tilde{P}\tilde{G}^* = O_{1 \times n^*}$. Then $x\tilde{P}(B^+(b))^* = 0$ for all $b \in \text{Null } T_0^* \oplus \{0\}$. It follows from (II)(i) that

$$x\tilde{P} = B(a)C$$

for some $a \in \text{Null } T_1 \oplus \{0\}$. In particular,

$$P(B(a))^* = O_{m \times 1},$$

and so $a \in \text{Null } T \oplus \{0\}$. Now

$$V(a) = x$$

if, and only if,

$$B(a)\, C^{*-1}\tilde{P}^* = x\tilde{P}C^{-1}C^{*-1}\tilde{P}^*.$$

This is equivalent to

$$B(a)\, C^{*-1}\tilde{P}^* = B(a)\, C^{*-1}\tilde{P}^*$$

as $x\tilde{P}C^{-1} = B(a)$. This shows that V_T is onto.

Take any $y \in l_2^m$ such that $yPG^* = O_{1 \times n}$. Then as before, using (II)(i),

$$yP = B^+(b)\, C^*$$

for some $b \in \text{Null } T_0^* \oplus \{0\}$. This b belongs necessarily to $\text{Null } T^* \oplus \{0\}$. However,

$$U^+(b) = B^+(b)\, C^*P^*(PP^*)^{-1} = yPP^*(PP^*)^{-1} = y.$$

This shows that $U_{T^*}^+$ is onto, and so (II)(i) implies (II)(ii).

(III) This is clear by the diagram. This completes the proof.

Remark 2.1. If Range T_1 is dense, and Range T_0^* is w^*-dense, then the finite-dimensional analogue of (II)(i) (that is, the case when l_2 is replaced by a finite-dimensional complex Euclidean space) is

$$\dim T_1/T_0 = \dim \operatorname{Null} T_1 + \dim \operatorname{Null} T_0^*$$

(see [10, Theorem 1.4]).

We do not know whether or not (III)(i) $\Rightarrow$ (II)(i). However, if this is true, then (III)(i), (III)(ii) and (III)(iii) are all equivalent as it is always true that (III)(iii) $\Rightarrow$ (III)(ii) $\Rightarrow$ (III)(i).

In the following theorem the existence of the solutions of boundary value problems subject to finite or infinite *nonhomogeneous* conditions is replaced by the problems of checking certain equations with the solutions to adjoint equations subject to finite or infinite *homogeneous* equations. This is an infinite-dimensional generalization of Theorem 1.1 of [10]. The problems occur naturally from ordinary differential operators if we allow admissible functions to be piecewise continuous at infinite points. We must point out, however, that the theorem below is not applicable *directly* to the problem: Suppose that S is a closed linear operator whose graph is contained in $X_1 \oplus X_2$. Let S^0 be a linear operator from the domain of S into X_2. Let x, g be given elements in X_2. Find y in the domain of S such that

$$Sy = g, \qquad \text{subject to} \quad S^0 y = x.$$

The condition $S^0 y = x$ is not a boundary condition in our sense because S^0 is not l_2^m-valued. However, if we set T to be the graph of the operator

$$y \to \{Sy, S^0 y\} \in X_2 \oplus X_2$$

for $y \in \operatorname{Domain} S$, then $T \subset X_1 \oplus (X_2 \oplus X_2)$, and the problem can be reduced to the problem in the theorem.

THEOREM 6. *Let P be a $m \times \infty$* normalized *Hilbert matrix, and $g \in \operatorname{Range} T_1$ and $\gamma \in l_2^m$ be given. Then we have the following*:

(I) *If there exists an element x in the domain of T_1 such that*

$$\{x, g\} \in T_1 \qquad \textit{and} \qquad P(B(\{x, g\}))^* = \gamma', \tag{NBP}$$

then

$$\overline{b_2(g)} = i\bar{\gamma}(PP^*)\ {}^{1}PC(B^+(b))^*$$

for all $\{b_2, 0\} \in T_0^*$, which satisfy one of the following three equivalent conditions:

(i) $\{b_2, 0\} \in T^*$,

where

$$T = \{a \in T_1 \mid P(B(a))^* = O_{m\times 1}\}.$$

(ii) $B^+(\{b_2, 0\})\, C^* \in \langle P\rangle$.

(iii) $(I_\infty - P^*(PP^*)^{-1}P)\, C(B^+(\{b_2, 0\}))^* = O_{\infty\times 1}$.

(II) *Assume that*

(i) $l_2 = \{B(a)C + B^+(b) \mid a \in \text{Null } T_1 \oplus \{0\},$
$b \in \text{Null } T_0^* \oplus \{0\}\}$,

(ii) $\{B(a)P^* \mid a \in \text{Null } T_1 \oplus \{0\}\}$ *is closed in* l_2^m.

Then the converse of (I) *remains valid.*

Proof. The proof, in particular, for (II), is quite different from the corresponding finite-dimensional case used in [10].

(I) Let $\tilde{P}$ be a $\tilde{m} \times \infty$ normalized Hilbert matrix such that

$$\langle \tilde{P}\rangle = l_2 \ominus \langle PC^{*-1}\rangle.$$

Then for $z \in l_2$, the following are equivalent:

$$(I_\infty - P^*(PP^*)^{-1}P)Cz^* = O_{\infty\times 1},$$
$$\tilde{P}z^* = O_{\tilde{m}\times 1}, \qquad z \in \langle PC^{*-1}\rangle.$$

Now take any $b \equiv \{b_2, 0\} \in T_0^*$ satisfying one of the three conditions (i), (ii), (iii) in (I). Then it follows from Theorem 1 that

$$\overline{b_2(g)} = i\bar{\gamma}(PP^*)^{-1}PC(B^+(b))^*.$$

(II) Let a_1 be an element in the domain of T_1 such that $a \equiv \{a_1, g\} \in T_1$. Then for all $\alpha \in l_2^n$,

$$a + \{\alpha\Phi^t, 0\} \in T_1.$$

The converse of (II) will be true if the equation

$$P(B(a))^* + P(B(\{\alpha\Phi^t, 0\}))^* = \gamma^t \tag{2.14}$$

has a solution $\alpha \in l_2^n$. This is true if, and only if,

$$\bar{\gamma} - B(a)\, P^* \in \text{Range}(\cdot GP^*). \tag{2.15}$$

Now

$$\text{Range}(\cdot GP^*) = \{B(a)\, P^* \mid a \in \text{Null } T_1 \oplus \{0\}\}$$

is closed by (II)(ii). It follows that (2.15) is equivalent to

$$\bar{\gamma}y^* = B(a)\,P^*y^*, \tag{2.16}$$

for all $y \in l_2^m$ such that $yPG^* = O_{1\times n}$.

Now for $y \in l_2^m$, $yPG^* = O_{1\times n}$ if, and only if, $yP(P(u))^* = 0$ for all $u \in \text{Null}\ T_1 \oplus \{0\}$. For such $y \in l_2^m$, by assumption (II)(i), there exists $b \in \text{Null}\ T_0^* \oplus \{0\}$ such that

$$yP = B^+(b)\,C^*,$$

and, in particular,

$$y = B^+(b)\,C^*P^*(PP^*)^{-1}.$$

This b belongs to Null $T^* \oplus \{0\}$, where

$$T = \{a \in T_1 \mid P(B(a))^* = O_{m\times 1}\}.$$

For,

$$\tilde{P}(B^+(b))^* = \tilde{P}C^{-1}P^*y^* = O_{m\times m}\,y^* = O_{\tilde{m}\times 1}.$$

Thus (2.16) is equivalent to

$$\bar{\gamma}(PP^*)^{-1}PC(B^+(b))^* = B(a)\,C(B^+(b))^* \tag{2.17}$$

for all $b \in \text{Null}\ T^* \oplus \{0\}$.

However, it is given that

$$\overline{b_2}(a_2) = i\bar{\gamma}(PP^*)^{-1}PC(B^+(b))^*$$

for all $\{b_2, 0\} \in T^*$.

It follows that (2.17) is equivalent to

$$-i\overline{b_2}(g) = B(a)\,C(B^+(b))^*$$

for $b = \{b_2, 0\} \in \text{Null}\ T^* \oplus \{0\}$.

However, this is always true by Theorem 1. Hence the converse of (I) is true. This completes the proof.

Remark 2.2. Range $(\cdot P^*)$ is closed if P is normalized. If Range T_0^* is w^*-dense, then

$$\{B(a) \mid a \in \text{Null}\ T_1 \oplus \{0\}\}$$

is closed. If dim Null $T_1 < \infty$ or Rank $P < \infty$, then

$$\text{Range}(\cdot GP^*) = \{B(a)P^* \mid a \in \text{Null}\ T_1 \oplus \{0\}\}$$

is closed. However, if dim Null $T_1 = \text{Rank } P = \infty$, then this set may not be closed even when dim Null $T_0 = 0$. For a concrete example, see Remark 3.1.

Remark 2.3. In the above theorem, (NBP) has a unique solution if dim Null $T^* = 0$, and (i) and (ii) of (II) are satisfied, where

$$T^* = \{b \in T_0^* \mid B^+(b)\, C^* \in \langle P \rangle\}.$$

The following is the dual version of the previous theorem. Notice the way C enters into the theorem.

THEOREM 7. *Let $\tilde{P}$ be a $\tilde{m} \times \infty$* normalized *Hilbert matrix. Let $g \in$* Range T_0^* and *$\tilde{\gamma} \in l_2^{\tilde{m}}$ be given. Then we have the following*:

(I) *If there exists an element x in the domain of T_0^* such that*

$$\{x, g\} \in T_0^* \qquad \textit{and} \qquad \tilde{P}(B^+(\{x, g\}))^* = \tilde{\gamma}^t, \qquad \text{(NBP)}^+$$

then

$$g(a_1) = i\bar{\tilde{\gamma}}(\tilde{P}\tilde{P}^*)^{-1}\tilde{P}C^*(B(\{a_1, 0\}))^*,$$

for all $\{a, 0\} \in T_1$ which satisfy one of the following three equivalent conditions:

(i) $\{a_1, 0\} \in {}^*\tilde{T}$,

where

$$\tilde{T} = \{b \in T_1^+ \mid \tilde{P}(B^+(b))^* = O_{\tilde{m} \times 1}\}.$$

(ii) $B(\{a_1, 0\})C \in \langle P \rangle$.

(iii) $(I_\infty - \tilde{P}^*(\tilde{P}\tilde{P}^*)^{-1}\tilde{P})\, C^*(B(\{a_1, 0\}))^* = O_{\infty \times 1}$.

(II) *Assume that*

(i) $l_2 = \{B(a)C + B^+(b) \mid a \in \text{Null } T_1 \oplus \{0\},$
$\qquad b \in \text{Null } T_0^* \oplus \{0\}\}$,

(ii) $\{B^+(b)\,\tilde{P}^* \mid b \in \text{Null } T_0^* \oplus \{0\}\} = \text{Range}(\cdot \tilde{G}\tilde{P}^*)$ *is closed in $l_2^{\tilde{m}}$. Then the converse of* (I) *remains valid.*

3. AN APPLICATION

The theory developed in the previous section will be useful when we have intimate knowledge of the internal structure of T_1, T_0^*, and boundary operators B, and B^+ and the explicit form of C. In this section we will consider a second-order system of ordinary differential equation. We will divide an interval into infinitely many pieces, and admissible functions will be piecewise continuous, and apply this to a limit-point case at ∞. It is hoped that the theory will find its application to partial differential operators and control theory.

Let $\{(a_j, b_j) : j \in N\}$ be a set of disjoint, bounded, open intervals such that for some M_1, M_2

$$0 < M_1 \leqslant b_j - a_j \leqslant M_2 < \infty, \qquad \text{all} \quad j \in N. \tag{3.1}$$

Let q be a $r \times r$ complex-valued matrix function of $x \in I$, where I is the union of $[a_j, b_j]$ over N, such that if we denote the kth column of q by q_k, then for some M_3,

$$\int_{a_j}^{b_j} q_k^*(x)\, q_k(x)\, dx \leqslant M_3 < \infty, \qquad \text{for all } j \in N,\ k = 1, 2, \ldots, r. \tag{3.2}$$

Denote by $\mathscr{L}_2(I)$ the Hilbert space of $r \times 1$ matrix-valued functions y of $x \in I$ with $\int_I y^*(x)\, y(x)\, dx < \infty$. Define two closed linear manifolds T_0 and T_1 in $\mathscr{L}_2(I) \oplus \mathscr{L}_2(I)$ by

$$\begin{aligned} T_1 = \{\{y, y'' + qy\} \mid y \in \mathscr{L}_2(I);\ & y \in \mathscr{C}^1(a_j, b_j), \\ & y' \in AC_{\mathrm{loc}}(a_j, b_j), \text{ all } j \in N;\ y'' + qy \in \mathscr{L}_2(I)\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} T_0 = \{\{y, y'' + qy\} \in T_1 \mid\ & y^{(k)}(a_{j+}) = y^{(k)}(b_{j-}) = O_{r \times 1}, \\ & k = 0, 1;\ j \in N\}. \end{aligned} \tag{3.4}$$

Let $B(y)$ be the $1 \times \infty$ constant matrix defined by

$$B(y) = [(y(a_{j+}))^t, (y'(a_{j+}))^t, (y(b_{j-}))^t, (y'(b_{j-}))^t] \tag{3.5}$$

where j runs through N.

Then B restricted to Domain T_1 deffines a T_1-bounded linear operator onto l_2 whose kernel is T_0, and B restricted to Domain T_0^* defines a T_0^*-bounded linear operator onto l_2 whose kernel is T_1^*. For this see page 307 of [7], where the case $r = 1$, q is real-valued was proved, but the proof carries over without difficulty to the present case.

For $y \in \mathscr{L}_2(I)$, $z \in (\mathscr{L}_2(I))^*$, the pairing $\overline{z(y)} = (y, z)$ on $\mathscr{L}_2(I) \oplus (\mathscr{L}_2(I))^*$ is defined by

$$\int_I z^*(x)\, y(x)\, dx,$$

and the corresponding B^+ is defined by

$$B^+(y) = B(y), \qquad y \in \text{Domain } T_0^*.$$

Then B and B^+ are related by

$$\int_I (z^*(x)(y'' + q(x)y) - (z'' + q^*(x)z)^*y)\,dx = iB(y)\,C(B(z))^*$$

for all $y \in \text{Domain } T_1$, $z \in \text{Domain } T_0^*$.

Here C is the $\infty \times \infty$ self-adjoint, unitary matrix by

$$C = i \bigoplus^{\infty} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.6}$$

where 0 and 1 denote $O_{r\times r}$ and I_r, respectively.

We can check easily that

$$l_2 = \{B(y)C + B(z) \mid y \in \text{Null } T_1, z \in \text{Null } T_0^*\},$$

and that T_1 and T_0^* have dense ranges.

Theorems 5 and 6 now take the following form:

PROPOSITION 8. *Let T_1, T_0, B, C be as* (3.3)–(3.6). *Let P be a $m \times \infty$ normalized Hilbert matrix. Assume that* (3.1), (3.2) *holds. Then we have the following*:

(I) *Let*

$$T = \{\{y, y'' + qy\} \in T_1 \mid P(B(y))^* = O_{m\times 1}\},$$
$$\tilde{m} = \dim(l_2 \ominus \langle PC\rangle).$$

Then

$$\dim \text{Null } T \geqslant \tilde{m}, \qquad \dim \text{Null } T^* \leqslant m,$$
$$\dim \text{Null } T^* + \tilde{m} = \dim \text{Null } T + m = \infty.$$

(II) *Let $g \in \text{Range } T_1$ and $\gamma \in l_2^m$ be given. If there exists y in the domain of T_1 such that*

$$y'' + qy = g, \qquad P(B(y))^* = \gamma^t,$$

then

$$\int_I z^*g\,dx = i\bar{\gamma}(PP^*)^{-1}PC(B(z))^*$$

for all $z \in$ Domain T_0^* *such that*

$$z'' + q^* z = O_{r\times 1}, \qquad B(z)C \in \langle P \rangle.$$

If we assume further that

$$\{B(y)\,P^* \mid y \in \text{Null } T_1\}$$

is closed in l_2^m, *then the converse is also true.*

Let P_0 be the $\infty \times \infty$ matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \left(\bigoplus^{\infty} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \right), \tag{3.7}$$

where the entries 0 and 1 denote the $O_{r\times r}$ and I_r, respectively. Let

$$\gamma = [\gamma_1, \gamma_2, \ldots],$$

where each γ_j is $1 \times r$ constant matrix. Assume that

$$b_j = a_{j+1} \qquad \text{for all } j \in \mathbb{N}.$$

Then

$$P_0(B(y)^* = \gamma^{\mathrm{t}}$$

if, and only if,

$$y(a_j +) = \gamma_{2j-1}^*, \qquad y'(a_j +) = \gamma_{2j-1}^*$$

for all $j \in N$, and

$$B(z)C \in \langle P_0 \rangle$$

if, and only if,

$$z^{(k)}(a_{j+1} +) = z^{(k)}(a_{j+1} -) \qquad \text{for all } j \in N.$$

If we allow that $\gamma_j = O_{1\times r}$ for $j = 3, 4, \ldots$, then the above proposition has the following form:

COROLLARY 9. *Assume* (3.1), (3.2), *and that* $b_j = a_{j+1}$ *for all* $j \in N$. *Let* $g \in$ Range T_1 *and let* γ_1, γ_2 *be* $1 \times r$ *constant matrices.*

If there exists y *in* $\mathscr{L}_2[a_1, \infty)$ *such that*

$$\begin{gathered} y \in \mathscr{C}^1[a_1, \infty); \qquad y' \in AC_{\mathrm{loc}}[a_1, \infty); \\ y'' + q(x)\,y = g \qquad \text{in } \mathscr{L}_2[a_1, \infty);\ y(a_1) = \gamma_1^*,\ y'(a_1) = \gamma_2^*, \end{gathered} \tag{3.8}$$

then

$$\int_{a_1}^{\infty} z^* g\,dx = \overline{\gamma_1 z'(a_1)} - \overline{\gamma_2 z(a_1)}$$

for all $z \in \mathscr{L}_2[a_1, \infty)$ *such that*

$$z \in \mathscr{C}^1[a_1, \infty), \qquad z' \in AC_{\mathrm{loc}}[a_1, \infty), \qquad z'' + q^*(x) z = O_{r \times 1}. \quad (3.9)$$

The converse remains valid if we assume further that

$$\{B(y)\, P_0^* \mid y \in \text{Domain } T_1\}$$

is closed in l_2, *where* P_0 *is defined in* (3.7).

Remark 3.1. If $q = 0$, and $b_j = a_{j+1}$ for all $j \in N$, then the set

$$\{B(y)\, P_0^* \mid \text{Domain } T_1\}$$

in the above is not closed in l_2. For, if it were, then, since

$$\{z \in \mathscr{L}_2[a_1, \infty) \mid z \in \mathscr{C}^1[a_1, \infty), z' \in AC_{\mathrm{loc}}[a_1, \infty), z'' = O_{r \times 1}\}$$

is the trivial space, it follows from the above proposition (or by imitating the proof of (II) of Theorem 6) that there exists $y \in \mathscr{L}_2[a_1, \infty)$ such that $y \in \mathscr{C}^1[a_1, \infty)$, $y \in AC_{\mathrm{loc}}[a_1, \infty)$, $y'' = O_{r \times 1}$, $y(a_1) = I_r$, $y(a_1) = O_{r \times r}$. This is impossible.

The following demonstrates that the theory developed so far can be applied to the deficiency index theory for ordinary differential operator. This is done by subdividing an interval into infinite intervals and by treating a continuity at a point as a boundary condition. Such an idea seems new in the literature.

THEOREM 10. *Let* $r(x)$ *be a real-valued function on* $[a_1, \infty)$ $(|a_1| < \infty)$. *Suppose that there exists a sequence of numbers* $a_1 < a_2 < \cdots < a_j < \cdots$ *such that for some positive numbers* M_1, M_2, M_3,

$$0 < M_1 \leqslant a_{j+1} - a_j \leqslant M_2 < \infty,$$

$$\int_{a_j}^{a_{j+1}} r^2(x)\,dx \leqslant M_3 < \infty,$$

for all $j \in N$.

Let $\mathscr{M}$ *be the vector space of* $1 \times \infty$ *constant vectors (necessarily contained in* l_2*) of the form*

$$\left[y(a_1+), y'(a_1+), \frac{y(a_2-)-y(a_2+)}{\sqrt{2}}, \frac{y'(a_2-)-y'(a_2+)}{\sqrt{2}}, \ldots, \frac{y(a_j-)-y(a_j+)}{\sqrt{2}}, \frac{y'(a_j-)-y'(a_j+)}{\sqrt{2}}, \ldots \right]$$

where y runs through the space of all $y \in \mathscr{L}_2[a_1, \infty)$ such that $y \in \mathscr{C}^1(a_j, a_{j+1})$, $y' \in AC_{\mathrm{loc}}[a_j, a_{j+1})$ for all $j \in N$, and $y'' + r(x) y \in \mathscr{L}_2[a_1, \infty)$.

If $\mathscr{M}$ is closed in l_2, then for each non-real complex number λ, there exists only one linearly independent function $y \in \mathscr{L}_2[a_1, \infty)$ such that $y \in \mathscr{C}^1[a_1, \infty)$, $y' \in AC_{\mathrm{loc}}[a_1, \infty)$, and $y'' + r(x) y = \lambda y$ for a.a. $x \in [a_1, \infty)$.

Proof. In Corollary 9, replace $q(x)$ by $r(x) - \lambda$ and g by 0. The set $\mathscr{M}$ is the same as

$$\{B(y) P_0^* ; y \in \text{Domain } T_1\}.$$

Suppose by contrary that the conclusion is false. Then $y'' + r(s) y$ is a limit-circle case at $+\infty$. It follows that for any given complex numbers γ_1 and γ_2 there exists a function $y \in \mathscr{L}_2[a_1, \infty)$ satisfying (3.8) with $q(x)$ being replaced by $r(x) - \lambda$ and g by 0.

By Corollary 9,

$$\gamma_1 z'(a_1) = \gamma_2 z(a_1)$$

for all $z \in \mathscr{L}_2[a_1, \infty)$ satisfying (3.9) with $q^*(x)$ being replaced by $r(x) - \bar{\lambda}$. However, the dimension of the space of such z is two. It follows that the value of z and z' at a_1 can be arbitrary, and so $\gamma_1 = \gamma_2 = 0$. This is a contradiction.

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